1. A continuous uniform distribution on the interval [0, k] has mean $\frac{k}{2}$ and variance $\frac{k^2}{12}$. A random sample of three independent variables X_1, X_2 and X_3 is taken from this distribution.

(a) Show that $\frac{2}{3}X_1 + \frac{1}{2}X_2 + \frac{5}{6}X_3$ is an unbiased estimator for k.

(3)

An unbiased estimator for k is given by $\hat{k} = aX_1 + bX_2$ where a and b are constants.

(b) Show that Var $(\hat{k}) = (a^2 - 2a + 2)\frac{k^2}{6}$

(6)

(c) Hence determine the value of a and the value of b for which $\hat{k}$ has minimum variance, and calculate this minimum variance.

(6)

(Total 15 marks)

1.	(a)	$\mathrm{E}\left(\frac{2}{3}X_1 + \frac{1}{2}X_2 + \frac{5}{6}X_3\right) = \frac{2}{3}\times\frac{k}{2} + \frac{1}{2}\times\frac{k}{2} + \frac{5}{6}\times\frac{k}{2} = k$	M1 A1	
		$\mathrm{E}(X_1 + X_2 + X_3) = k \Rightarrow$ **unbiased**	B1	3
	(b)	$\mathrm{E}(aX_1 + bX_2) = a\frac{k}{2} + b\frac{k}{2} = k$	M1	
		$a + b = 2$	A1	
		$\mathrm{Var}(aX_1 + bX_2) = a^2\frac{k^2}{12} + b^2\frac{k^2}{12}$	M1A1	
		$= a^2\frac{k^2}{12} + (2-a)^2\frac{k^2}{12}$	M1	
		$= (2a^2 - 4a + 4)\frac{k^2}{12}$		
		$= (2a^2 - 2a + 4)\frac{k^2}{6}$ (*) since answer given	A1 cso	6
	(c)	Min value when $(2a-2)\frac{k^2}{6} = 0$		
		$\frac{\mathrm{d}}{\mathrm{da}}(\mathrm{Var}) = 0$, all correct, condone missing $\frac{k^2}{6}$	M1A1	
		$\Rightarrow 2a - 2 = 0$	A1A1	
		$A = 1$, $b = 1$.		
		$\frac{\mathrm{d}^2(\mathrm{Var})}{\mathrm{da}^2} = \frac{2k^2}{6 > 0}$ since $k^2 > 0$ therefore it is a minimum	M1	
		min variance $= (1 - 2 + 2)\frac{k^2}{6}$		
		$= \frac{k^2}{6}$	B1	6
		<u>Alternative</u>		
		$\frac{k^2}{6}(a-1)^2 - \frac{k^2}{6} + \frac{2k^2}{6}$	M1 A1	
		$\frac{k^2}{6}(a-1)^2 + \frac{k^2}{6}$	M1	
		Min when $\frac{k^2}{6}(a-1)^2 = 0$	A1A1	
		$a = 1$ $b = 1$		
		min var $= k^2/6$	B1	

[15]

1. This question proved to be the most challenging for many candidates. In part (a) many candidates tried to prove it was equal to $\frac{k}{2}$ and few made the concluding statement that it was unbiased.

In part (b) few candidates were able to find $a + b = 2$ and hence made little progress. Those who did find this were able to gain full marks.

In part (c) a mix of both of the given methods on the mark scheme were used. If they chose the first method the majority of candidates did not prove that it was a minimum. If they chose the second they rarely completed the square correctly choosing to leave out the $\frac{k^2}{6}$.